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LETTER TO THE EDITOR

Recovery of spontaneously broken supersymmetry in quantum mechanics

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Abstract. The connection between quantum mechanical systems with broken and unbroken supersymmetry is discussed The class of superpotentials is given for which it is possible to find eigenvalues and eigenfunctions by recovering the broken supersymmetry

It is known that the supersymmetry of a quantum mechanical system enables one to find easily the energy spectrum and the wavefunctions of this system by building up the hierarchy of Hamiltonians [1–3]. In this case the existence of a ground (non-degenerate) state with zero energy in a system (i.e. the supersymmetry unbreaking) plays a crucial role For the broken supersymmetry the methods allowing an easy determination of wavefunctions and spectrum of a system are now absent. We are aware of only one publication [4] where an attempt to study this problem has been made. In this letter we find a class of superpotentials for which it proves to be possible to 'recover' the spontaneously broken supersymmetry.

In the first papers on supersymmetric quantum mechanics [5] it was shown that the Schrödinger equation for the one-dimensional supersymmetric quantum mechanical system can be factorized in the following way:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+W(x)\right)f(x)=kb(x)$$
$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+W(x)\right)b(x)=kf(x) \tag{1}$$

where $W(x)$ is the superpotential and $f(x)$ and $b(x)$ are the wavefunctions of the 'fermionic' and 'bosonic' sectors of the problem. The case $k=(2E)^{1/2}>0$ corresponds to the broken supersymmetry. Further, we consider the discrete spectrum of the problem, the wavefunctions being normalized according to the condition

$$\int \mathrm{d}x([f(x)]^2+[b(x)]^2)=1.$$

Now we are searching for a non-degenerate linear transformation connecting the eigenfunctions of the initial system with the eigenfunctions of another supersymmetric system

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}+V(x)\right)F(x)=k_1B(x)$$
$$\left(-\frac{\mathrm{d}}{\mathrm{d}x}+V(x)\right)B(x)=k_2F(x) \tag{2}$$

with a new superpotential $V(x)$ such that there exists a normalizable solution of one of the two equations (2) for $k_1 = 0$ and/or $k_2 = 0$. In other words, supersymmetry is unbroken in this system. Let the transformation we are looking for be given by the matrix $\hat{A}$:

$$\begin{pmatrix} f(x) \\ b(x) \end{pmatrix} = \hat{A} \begin{pmatrix} F(x) \\ B(x) \end{pmatrix} \equiv \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix} \begin{pmatrix} F(x) \\ B(x) \end{pmatrix}. \tag{3}$$

Substituting (3) into system (1) gives rise to a system:

$$\begin{aligned} &(\alpha\beta - \gamma\delta)F' + [\alpha'\delta - \gamma'\beta - k(\alpha\beta + \gamma\delta) + W(\alpha\delta + \beta\gamma)]F \\ &\qquad = [\beta\delta' - \beta'\delta + k(\beta^2 + \delta^2) - 2W\beta\delta]B \\ &-(\alpha\beta - \gamma\delta)B' + [\beta'\gamma - \alpha\delta' - k(\alpha\beta + \gamma\delta) + W(\alpha\delta + \beta\gamma)]B \\ &\qquad = [\alpha\gamma' - \alpha'\gamma + k(\alpha^2 + \gamma^2) - 2W\alpha\gamma]F \end{aligned} \tag{4}$$

(here and below, a prime means differentiation on x).

The equality of expressions in square brackets on the left-hand sides of equations (4), i.e. the existence of a superpotential, leads to the following constraint on the transformation matrix elements:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\alpha\delta - \beta\gamma) = 0 \tag{5}$$

i.e. the matrix determinant must be constant. Without loss of generality it may be set equal to unity. Note further that, since new eigenfunctions should obey the normalization condition

$$\int \mathrm{d}x([F(x)]^2 + [B(x)]^2) = 1$$

then it follows that $\hat{A} \in \mathrm{SO}(2)$. Therefore the transformation matrix can be parametrized as usual:

$$\hat{A} = \begin{pmatrix} \cos\phi(x) & \sin\phi(x) \\ -\sin\phi(x) & \cos\phi(x) \end{pmatrix}. \tag{6}$$

Now system (4) is rewritten as:

$$\begin{aligned} F' + W\cos(2\phi)F &= [k - \phi' - W\sin(2\phi)]B \\ -B' + W\cos(2\phi)B &= [k - \phi' + W\sin(2\phi)]F. \end{aligned} \tag{7}$$

The factors in square brackets on the right-hand sides of system (7) should be equal to constants k_1 and k_2:

$$\begin{aligned} k - \phi' - W\sin(2\phi) &= k_1 \\ k - \phi' + W\sin(2\phi) &= k_2. \end{aligned} \tag{8}$$

One can readily find from this system not only the function ϕ but also the initial superpotential W†:

$$W(x) = \frac{a}{\sin(bx + c)} \qquad \phi(x) = \tfrac{1}{2}(bx + c) \tag{9}$$

† Note that for $k_1 = k_2$ system (8) has no non-trivial solutions

where c is the arbitiary constant and the constants a and b are related to k, k_1 and k_2 via:

$$k_1 = k - a - b/2 \qquad k_2 = k + a - b/2. \tag{10}$$

The new superpotential is

$$V(x) = a \cot(bx + c).$$

It is obvious that for this superpotential there exists a normalizable eigenfunction corresponding to the zero-energy state which is the ground state Consequently, supersymmetry in this new system is unbroken. In addition, the potential is, as one can see, shape invariant [2] and the spectrum of the system can be obtained by constructing the hierarchy of Hamiltonians [3]. The spectrum of the initial system may be found by using equations (10), and the wavefunctions are obtained by transforming $F(x)$ and $B(x)$ by $\hat{A}$

The situation described above is realized, for example, in a problem on the electron angular momentum in the Pauli theory. In this case the problem of finding the common eigenfunctions of the squared angular momentum and of the projection of the angular momentum is reduced to the solution of the following system of first-order differential equations [6]:

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{1}{\sin\theta}(m+\tfrac{1}{2})\right) f(\theta) = kb(\theta) \\ &\left(-\frac{\mathrm{d}}{\mathrm{d}\theta} - \frac{1}{\sin\theta}(m+\tfrac{1}{2})\right) b(\theta) = kf(\theta) \end{aligned} \tag{11}$$

where m is the projection of the orbital angular momentum and k is the eigenvalue of the Dirac operator $\hat{K} = (\sigma L) + 1$. One can easily see that system (11) for spherical spinors corresponds to the general case studied above if one takes $a = m + \frac{1}{2}$, $b = 1$ and $c = 0$. The solution of this system along the lines of the method described above has been obtained in [7].

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